Black hole geometry & thermodynamics

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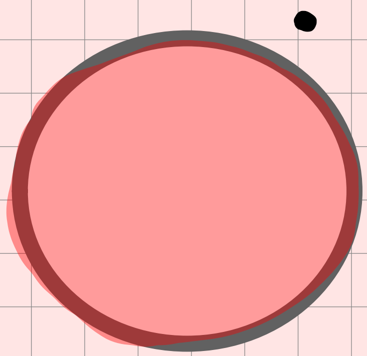
Basics of B.H:

* Heuristics
* Geometry
* BH Information Paradox



Star

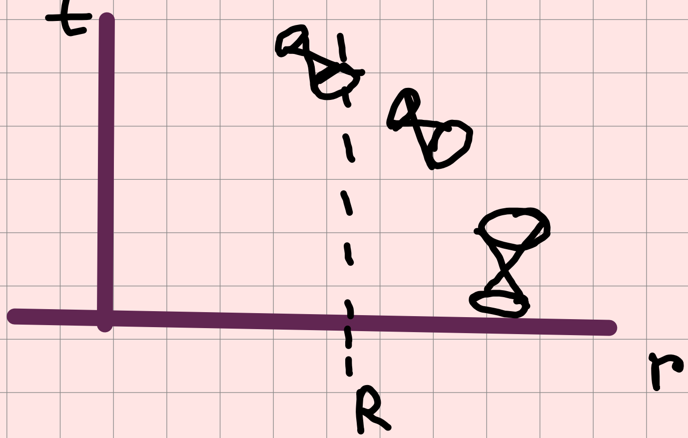
$$E = \frac{1}{2}mc^2 - \frac{GMm}{R} = 0$$

$$\frac{1}{2}mc^2 = \frac{GmM}{R}$$

$$\Rightarrow R = \frac{2GM}{c^2}$$

we used to call these dark stars (Light can't escape!)

Black holes!



* Things go in
  nothing comes out

Q. What does this mean quantum mechanically?

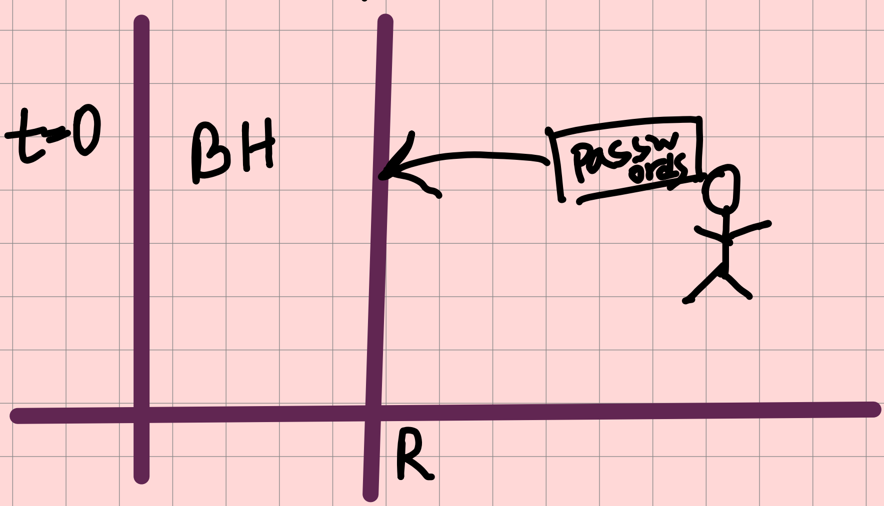
QM says information is preserved

$t=0 \quad |\psi\rangle \Rightarrow |\langle\psi|\psi\rangle|^2 = 1$

$t=T \quad U|\psi\rangle \Rightarrow$ time evolved

$$\boxed{\begin{aligned} &|\psi\rangle = \sum C_n|\psi_n\rangle \\ &|\langle\psi|U^{\dagger}U|\psi\rangle|^2 \\ &= |\langle\psi|\psi\rangle|^2 \\ &= 1 \end{aligned}}$$

Entropic point of view!



Entropy!

Entropy is a measure of what info is unknown.

The number of questions you have to ask to know the state.

Example: socks

1 label = $\begin{cases} \text{Right} \\ \text{Left} \end{cases}$

$P_R = P_L = \frac{1}{2}$

$S = -\sum P_i \log_2 P_i = -P_L \log_2 P_L - P_R \log_2 P_R$

ask 1 T/F ques.

⇓

1 bit of information

⇓

1

$= -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2}$

$= -2 \cdot \frac{1}{2}\log_2\frac{1}{2}$

$= 1$

$$\boxed{\text{Shannon Entropy} \\ = -\sum p(x) \log p(x)}$$

Socks : 2 Labels

$\begin{cases} R \\ L \end{cases}$ $\begin{cases} B \\ G \end{cases}$

$$P_{RB} = P_{RG} = P_{LB} = P_{LG} = \frac{1}{4}$$

$$S = -\frac{1}{4}\log_2\frac{1}{4} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{4}\log_2\frac{1}{4}$$
$$- \frac{1}{4}\log_2\frac{1}{4}$$
$$= -4 \cdot \frac{1}{4}\log_2\frac{1}{4}$$
$$= -\log_2\frac{1}{4} = 2$$

Passwords have N bits of info

Information uncertainty = N bits

$\Delta S = k_B N$

* Black holes increase entropy
* Entropy increase violates unitarity

Unitarity

$$|\langle\psi(t)|\psi(t)\rangle|^2 = |\langle\psi(0)|\psi(0)\rangle|^2$$

conflicts with QM

* First sign that gravity and QM have issues being roommates!

$E = pc$

* Entropy increase for BH suggests the existence of a temp

$$\Delta E = T \Delta S$$

A BH sitting alone in space shouldn't have a temp. It shouldn't rotate!?

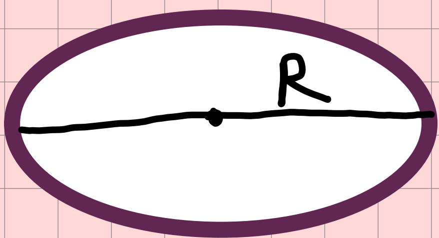
Guess the temp

$\Delta E = T\Delta S \qquad \Delta S = 1 \text{ bit}$

$\Delta E = T \cdot 1$

$$\Delta E \Delta t = \frac{\hbar}{2} \Rightarrow \Delta E = \frac{\hbar}{2\Delta t} = \frac{h}{8\pi GM}$$

$$\Delta t = 2 \cdot 2GM$$
$$= 4GM$$



Things which have a temp they radiate!

* BH must emit particles

$\Rightarrow$ BH's are quantum mechanical ?!

second link between QM & Gravity.

* Black holes are complicated systems and have thermodynamic properties.

$S \sim A$

Known facts

* surface gravity $\mathcal{K}$ is a constant on a event horizon.

acceleration by an observer at ∞ at the horizon

$$ds^2 = (\;)dt^2 + \frac{1}{(\;)}dr^2$$

* Entropy of a BH is proportional

to the area of the event horizon $S = \frac{A}{4G}$

Laws of thermo

0th law

existence of a quantitative measure that characterizes how things equilibrate with other bodies.

1st law

$$dE = TdS + PdV$$

2nd law

Entropy never decreases!

3rd law

Entropy → constant values as $T \to 0$

Laws of BH thermodynamics

0th law

surface gravity $\mathcal{K}$ is constant on the horizon.

1st law

$$dE = \frac{K}{8\pi} dA + \Omega dJ + \Phi dQ$$

angular momentum

2nd law

horizon area is non-decreasing

$$S = S_{BH} + S_{Radiation}$$

3rd law

can't set to $\mathcal{K} = 0$
by a finite number of steps

$$r^* = r + 2M \ln \frac{|r-2M|}{2M}$$

Black hole geometry

$M =$ mass of BH

$\kappa =$ surface gravity $=$ acceleration at the horizon as seen at ∞

$A =$ Area of event horizon

Schwarzschild BH

signature $- + + +$

$$ds^2 = \underbrace{-\left(1 - \frac{2M}{r}\right)}_{g_{00}} dt^2 + \underbrace{\frac{dr^2}{\left(1 - \frac{2M}{r}\right)}}_{g_{rr}} + r^2 d\Omega^2$$

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

$G = 1$, $c = 1$
[convention]

$g_{\mu\nu} dx^\mu dx^\nu$

note:

$r = 2M$

$g_{00} = 0$

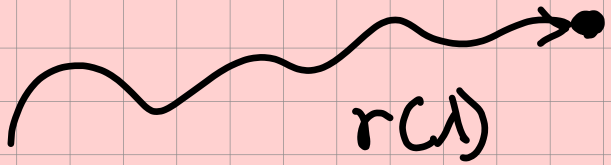
$g_{rr} = \infty$

coordinate singularity (not actual)

* $r = 0 \longrightarrow \underbrace{\text{actual}}_{\text{curvature}}$ singularity

$$R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} = \frac{12(2GM)^2}{r^6}$$

singularity



means when your geodesic kinda stops

Different coordinate systems:

a) Regge – Wheeler (Tortoise) coordinates

suppose you want to put the metric $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$

$$ds^2 = f(r)\left(-dt^2 + (dr^*)^2\right) + r^2 d\Omega^2$$

*Consider a radial null geodesic

$\hookrightarrow d\Omega = 0$

$ds^2 = 0$

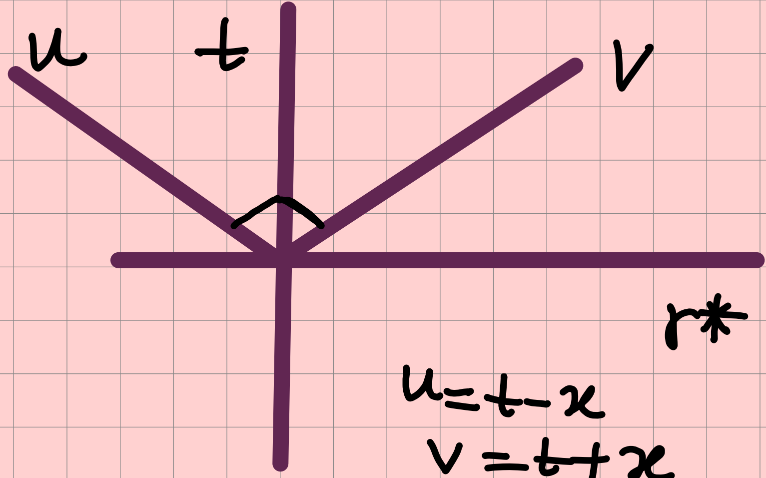
$\Rightarrow 0 = -\left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2$

$\Rightarrow dt^2 = \frac{1}{\left(1-\frac{2M}{r}\right)^2}dr^2 = (dr^*)^2$

$r^* = r + 2M \ln \frac{|r-2M|}{2M}$

$2M \leq r < \infty$

Usual trick



$v = t + r^*$

$u = t - r^*$

$u = t - x$
$v = t + x$
$dt^2 - dx^2 = du\,dv$

constant u, $t - r^* = \text{constant}$

$$t - r^* = 0$$

$$\Rightarrow t = r^*$$

$$ds^2 = \left(1 - \frac{2M}{r}\right)\Big(\underbrace{-dt^2 + (dr^*)^2}_{-du\,dv}\Big) + r^2 d\Omega^2$$

$$du = dv - 2dr^*$$

$$= \left(1 - \frac{2M}{r}\right)\Big(-(dv - 2dr^*)dv\Big) + r^2 d\Omega^2$$

$$= -\left(1 - \frac{2M}{r}\right)dv^2 - 2\left(1 - \frac{2M}{r}\right)dr^* dv + r^2 d\Omega^2$$

$$\frac{dr}{\left(1 - \frac{2M}{r}\right)}$$

$$ds^2 = -\left(1-\frac{2m}{r}\right)(dv)^2 + \left(1-\frac{2m}{r}\right)\cdot 2 \cdot \frac{1}{\left(1-\frac{2m}{r}\right)} dr\,dv + r^2 d\Omega^2$$

$$= -\left(1-\frac{2m}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega^2$$

want to show that you can pass through $r=2m$

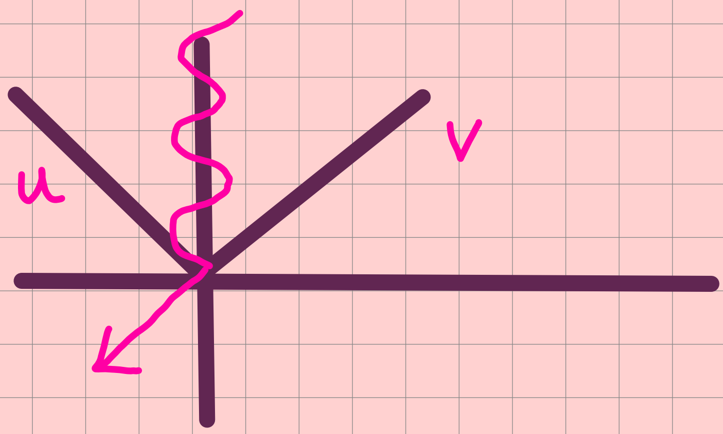
consider $r \le 2m$ [$d\Omega = 0$ as it is a radial geodesic]

$$2dr\,dv = ds^2 + \left(1-\frac{2m}{r}\right)dv^2$$

$$= -\left(\underbrace{-ds^2}_{\text{pos}} + \underbrace{\left(\frac{2m}{r}-1\right)}_{\text{pos}} dv^2\right)$$

* For a timelike geodesic $ds^2 \le 0$

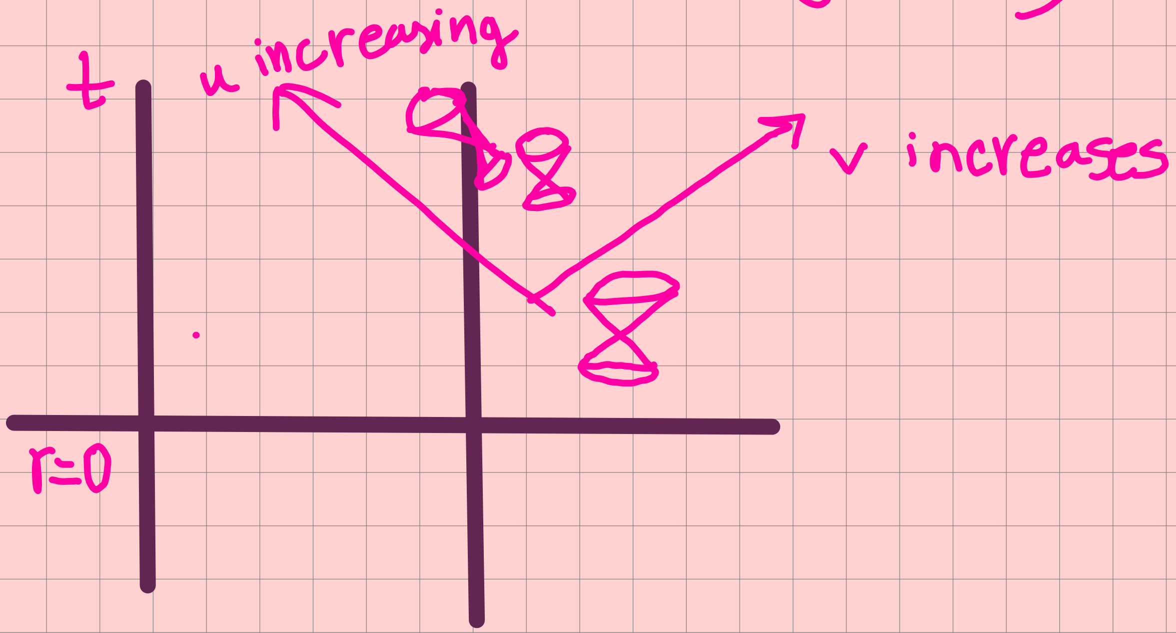
$$= -(\text{pos} + \text{pos}) \le 0$$

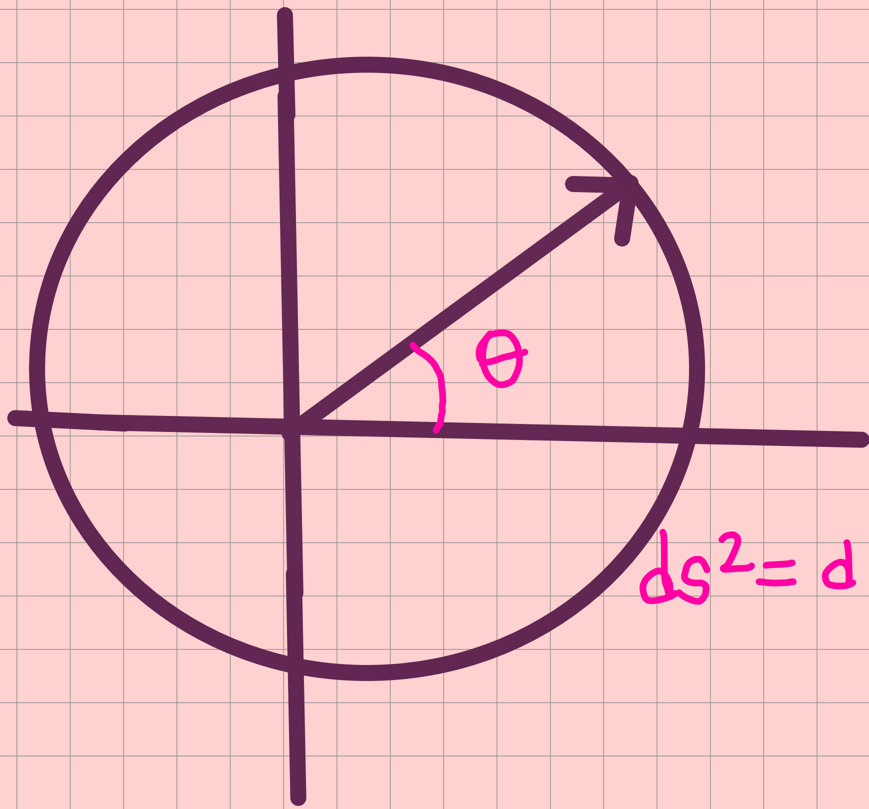


This means $dr \leq 0$

Conclusion:

geodesics can be extended beyond $r = 2M$. ($r = 2M$ is not a physical singularity. It is a coordinate singularity)





$$ds^2 = dr^2 + r^2 (d\theta)^2$$

KRUSCAL COORDINATES

$$ds^2 = \left(1 - \frac{2m}{r}\right) du\,dv + r^2 d\Omega^2$$

$$u = -e^{-u/4m} \qquad v = e^{\frac{v}{4m}}$$

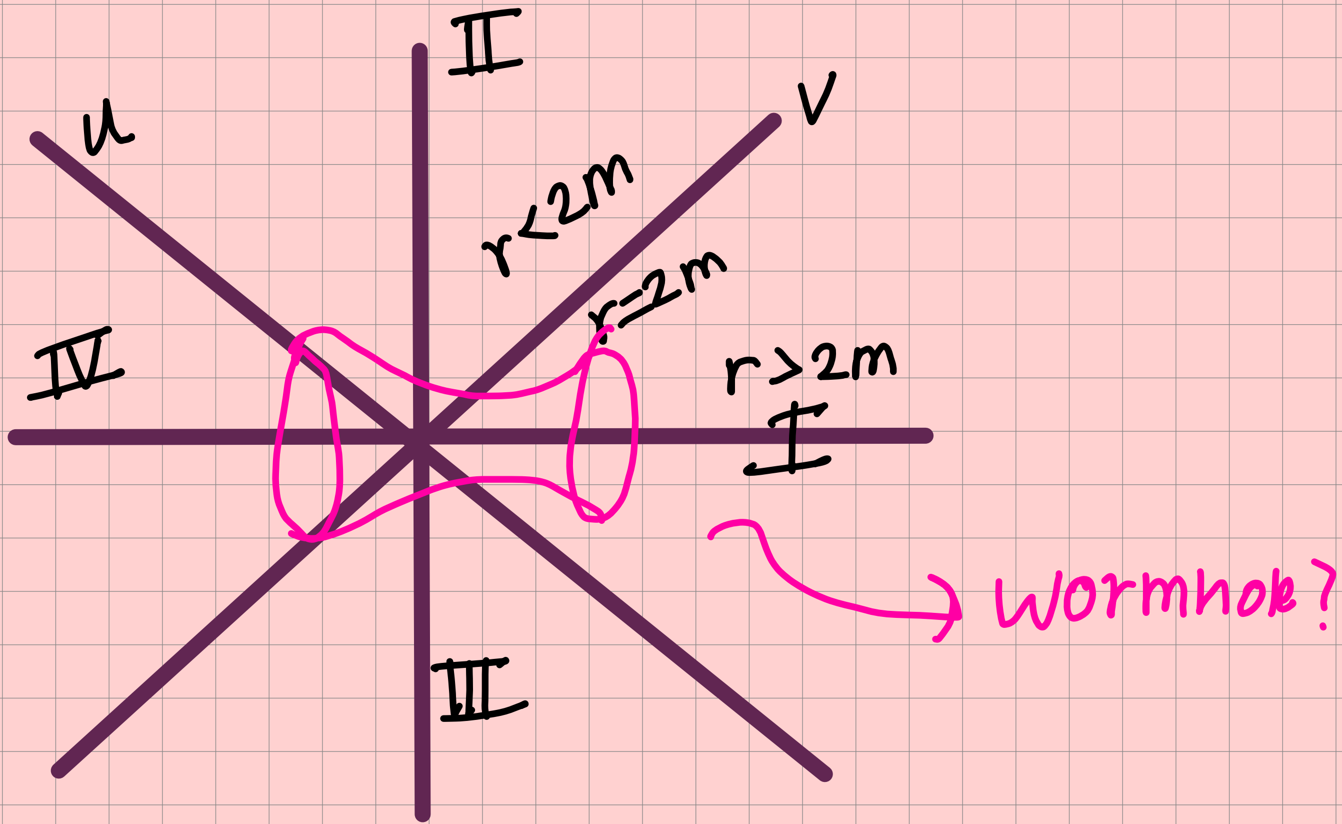
kruskal coordinates

u, v

$u(t, r^*), v(t, r^*)$

Eddington-Finkelstein coordinates

$$ds^2 = -\frac{32m^3}{r} e^{-r/2m} dv\,du + r^2 d\Omega^2$$



conformal compactification

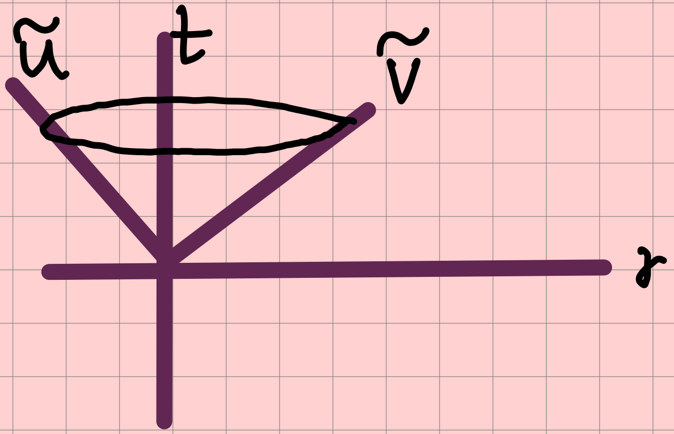
Carter - Penrose

* bringing the whole spacetime in a finite coordinate range

$$-\pi/2 < \tilde{u} < \frac{\pi}{2}$$

$$-\pi/2 < \tilde{v} < \frac{\pi}{2}$$

* extra regions from analytic continuation
* lines at $45°$ $\Rightarrow$ light rays $\Rightarrow$ null



$$L^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$L^2 = \lambda^2 \left((\tilde{x}_1 - \tilde{x}_2)^2 + (\tilde{y}_1 - \tilde{y}_2)^2 \right)$$

$$\tilde{L} = \frac{L^2}{\lambda^2} = (\tilde{x}_1 - \tilde{x}_2)^2 + (\tilde{y}_1 - \tilde{y}_2)^2$$

$$ds^2 = -\left(1-\frac{2m}{r}\right)du\,dv + r^2 d\Omega^2$$

$$u = \tan\tilde{u} \qquad du = \sec^2\tilde{u}\, d\tilde{u}$$

$$v = \tan\tilde{v} \qquad dv = \sec^2\tilde{v}\, d\tilde{v}$$

$$ds^2 = \frac{(\sec\tilde{u}\sec\tilde{v})^2}{2}\left(-4\left(1-\frac{2m}{r}\right)d\tilde{u}\,d\tilde{v} + r^2\cos^2\tilde{u}\cos^2\tilde{v}\, d\Omega^2\right)$$

$$r^* = \frac{1}{2}(v-u) = \frac{\sin(\tilde{v}-\tilde{u})}{2\cos\tilde{u}\cos\tilde{v}}$$

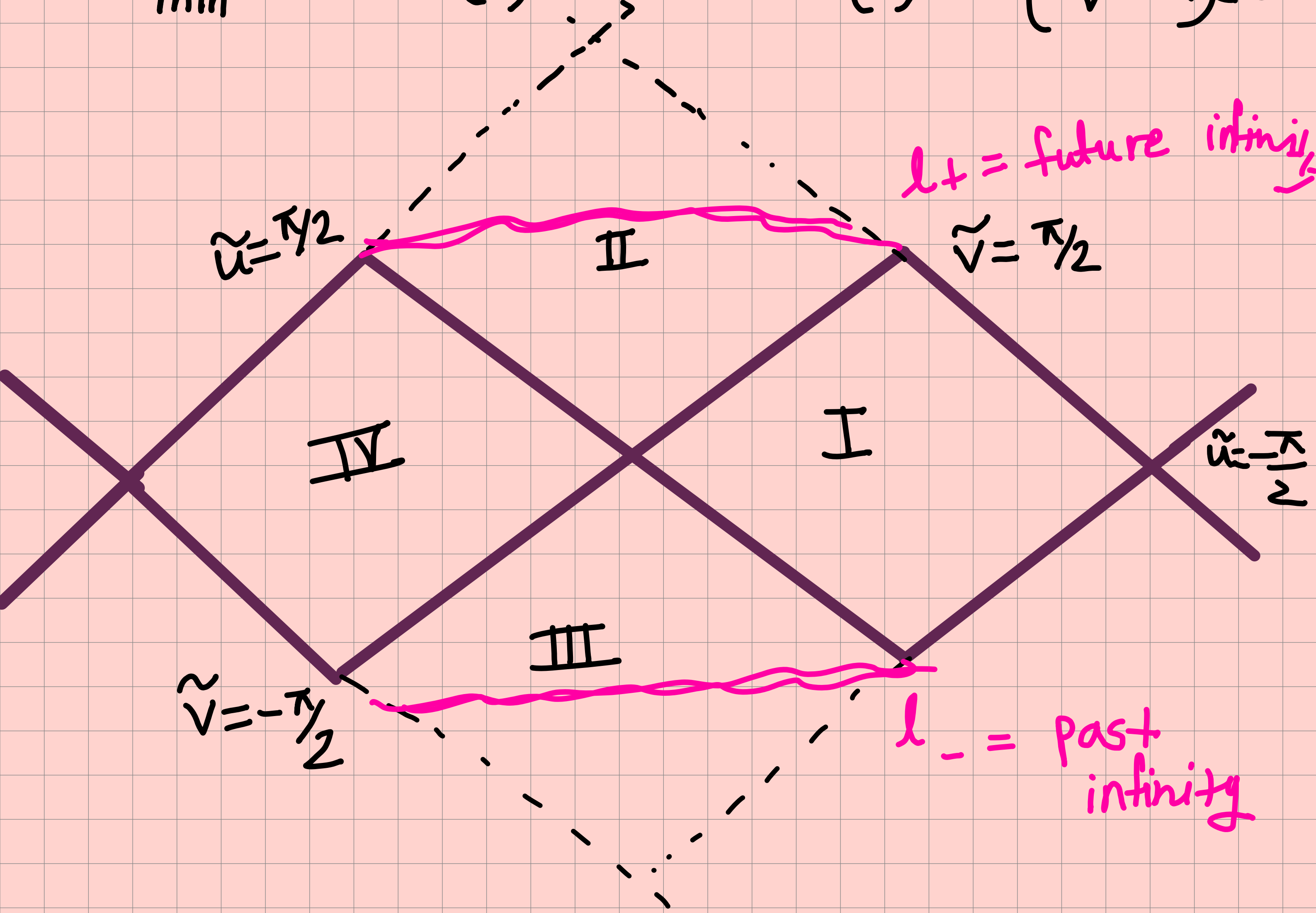
$$ds^2 = \Lambda^2 (d\tilde{s})^2$$

$$\Lambda^2 = \frac{1}{(2\cos\tilde{u}\cos\tilde{v})^2}$$

$$d\tilde{s}^2 = -4\left(1-\frac{2m}{r}\right)d\tilde{u}\,d\tilde{v} + \left(\frac{r}{r^*}\right)^2 \sin^2(\tilde{v}-\tilde{u})\, d\Omega^2$$

if I had been compactifying minkowski

$$d\tilde{s}^2_{min} = -4(\,)\,d\tilde{u}\,d\tilde{v} + (\,)\sin^2(\tilde{v}-\tilde{u})\,d\Omega^2$$



notes by naziee
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