Black holes in string theory

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Simplest solution

$$ds^2 = -c^2 dt^2 \left(1 - \frac{2GM}{rc^2}\right) + dr^2 \left(1 - \frac{2GM}{rc^2}\right)^{-1}$$
$$+ r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

G = newton's gravitational const.
c = speed of light
M = mass

Apparently singular at
$r = \frac{2GM}{c^2}$

coordinate singularity

if someone goes through the $r = \frac{2GM}{c^2}$ surface, there is no special effect.

Once we cross this, we cannot come back.

$\rightarrow r = \frac{2GM}{c^2}$ is the event horizon (one-way-membrane)

$r = \frac{2GM}{c^2}$ surface absorbs everything, emits nothing.

Black hole

Horizon area

$$r^2 \int d\theta \sin\theta \, d\phi \Big|_{r=\frac{2GM}{c^2}} = 4\pi \cdot \left(\frac{2GM}{c^2}\right)^2$$

quantum effects make black hole into a blackbody with finite temperature, entropy etc.

$$\underset{\text{entr}}{S} = \frac{k_B A c^3}{4G\hbar}$$

k_B = Boltzman const.

$\hbar$ = plank const.

$$S = \frac{k_B c^3}{4G\hbar} 4\pi \left(\frac{2GM}{c^2}\right)^2$$

$$dE = TdS \quad , \quad E = mc^2$$

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{1}{c^2}\frac{\partial S}{\partial M} = \frac{8\pi G}{c^3\hbar} k_B M$$

$$T = \frac{c^3\hbar}{8\pi G k_B M}$$

In stat mech

$$S = k_B \ln \Omega$$

$\Omega \rightarrow$ no of micro-states for a given set of macroscopic properties

Can we give a similar interpretation to black hole entropy?

- we need to construct the hilbert space & then count states with a given macroscopic property (Like M)

↳ this needs quantum theory of gravity

⇓

string theory

String theory has many vacua

we can ask questions about black hole entropy in any of the minima.

Strategy : understand black hole microstates in simple black holes in simple minima

Schwarzchlid is not the most convenient black hole.

$$T = \frac{c^3 \hbar}{8\pi G k_B M}$$

we can avoid this by considering chared blackholes

We'll set $c = 1, \hbar = 1, \underbrace{k_B = 1}$

what we'll call S is $\frac{S}{k_B}$

what we'll call T' is $k_B T$

$\hbar, c$ can be restored by dimensional analysis.

Actual expression =
derived expression $\times \hbar^{\alpha} c^{\beta}$

Fix α, β using dimension analysis

$$S = \frac{A}{4G}$$

$$\underbrace{\frac{S}{k_B}}_{\substack{\ln\Omega \\ \text{dimension less}}} = \frac{A}{4G} \hbar^{\alpha} \cdot c^{\beta}$$

$$\text{FORCE} = \frac{G m_1 m_2}{r^2}$$

$$[G] = [\text{FORCE}] \cdot L^2 \cdot M^{-2}$$

$$= MLT^{-2} \cdot L^2 M^{-2}$$

$$= M^{-1} L^3 T^{-2}$$

$$\left[\frac{S}{k_B}\right] = L^2 \overbrace{M L^{-3} T^2}^{1/G} (L M L T^{-1})^{\alpha} (L T^{-1})^{\beta}$$

$$\left.\begin{aligned} L &: 2 - 3 + \alpha + \beta = 0 \\ T &: 2 - \alpha - \beta = 0 \\ M &: 1 + \alpha = 0 \end{aligned}\right\} \quad \begin{aligned} \alpha &= -1 \\ \beta &= -3 \end{aligned}$$

$$S = k_B \frac{A}{4G} \hbar^{-1} c^3$$

now,

set $\hbar = 1$, $c = 1$, $k_B = 1$

Einstein - maxwell

$$\int d^4x \sqrt{-\det g} \left[\frac{1}{16\pi G} R - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$\partial_n = \frac{\partial}{\partial x^n}$$

This has a charged blackhole solution.

$$ds^2 = -dt^2 \left(1 - \frac{a}{r}\right)\left(1 - \frac{b}{r}\right) + \frac{dr^2}{\left(1-\frac{a}{r}\right)\left(1-\frac{b}{r}\right)} + r^2 (d\theta^2 + \sin^2\theta \, d\phi)$$

$$F_{rt} = \frac{q}{r^2}, \quad a+b = 2GM, \quad ab = 4\pi G q^2$$

$$a = GM + \sqrt{G^2M^2 - 4\pi G q^2}$$

$$b = GM - \sqrt{G^2M^2 - 4\pi G q^2}$$

$r = a$: outer horizon

$r = b$: inner horizon

not relevant for us

$$S = \frac{A}{4G}$$ → outer horizon area

$$= \frac{1}{4G} \times 4\pi a^2$$

$$= \frac{\pi}{G}\left\{GM + \sqrt{G^2M^2 - 4\pi G q^2}\right\}^2$$

$$\frac{1}{T} = \frac{\partial S}{\partial M} = \frac{2\pi}{G}\left(GM + \sqrt{G^2M^2 - 4\pi G q^2}\right)\left(G + \frac{G^2M}{\sqrt{G^2M^2 - 4\pi G q^2}}\right)$$

need $M^2 > 4\pi G^{-1} q^2$ for the horizon to exist.

extremal limit:

$$M^2 \to 4\pi G q^2 \,, \quad \frac{1}{T} \to \infty$$

$$T \to 0$$

supersymmetry $(N \geq 2)$

mass of any elementary or composite particle satisfies a bound.

$$m > \sqrt{\frac{4\pi}{G}}\, q$$

BPS bound

$\Downarrow$

ideal candidate for testing $S = \ln \Omega$

certain flat direction of V exist and control G, electromagnetic coupling etc.

controls the minimum unit of Charge q_0

$q = Nq_0$; N = integer

horizon radius $a = GM$

$$= \sqrt{\frac{4\pi}{G}}\, q G$$

$$= \sqrt{4\pi G}\, N q_0$$

$\frac{1}{M}$ = length Scale associated with quantum effects

($c = 1$, $\hbar = 1$)

quantum effects imply that you cannot localize the particle over distance $< \frac{1}{M} \xrightarrow{\text{extremal black hole}}$

$$\sqrt{\frac{G}{4\pi}}\,\frac{1}{q} = \sqrt{\frac{G}{4\pi}}\,\frac{1}{Nq_0}$$

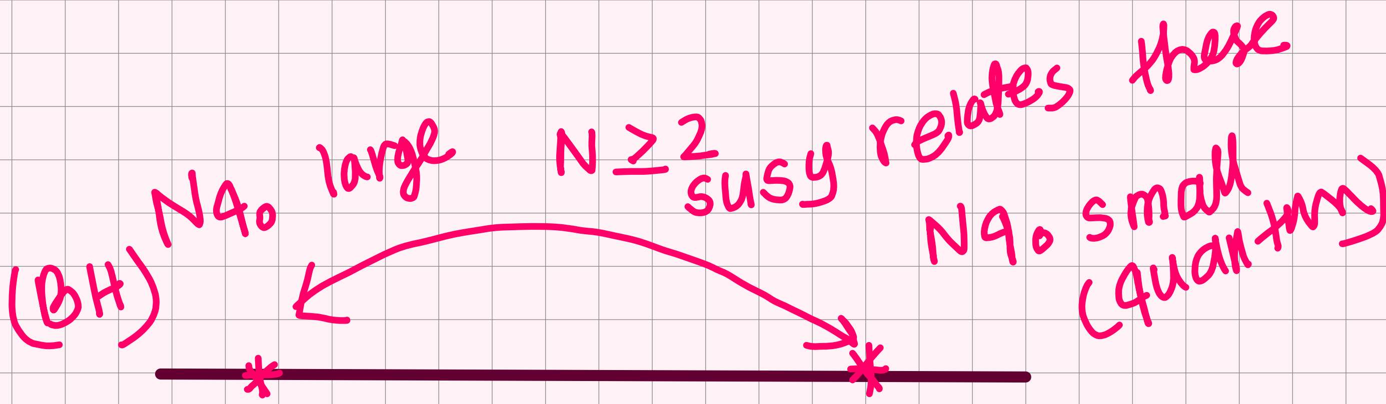
for a period $> \frac{1}{M}$

$Nq_0 >> 1$, the black hole description is good.

$Nq_0 << 1$, we can describe the system as a weakly interacting quantum system of N particles

$\Downarrow$

we can hope to count microstates



$Q = Nq_0$

minimal unit of charge

Nq_0 large $\rightarrow$ blackhole desuiption is good

$S = \frac{A}{4}$

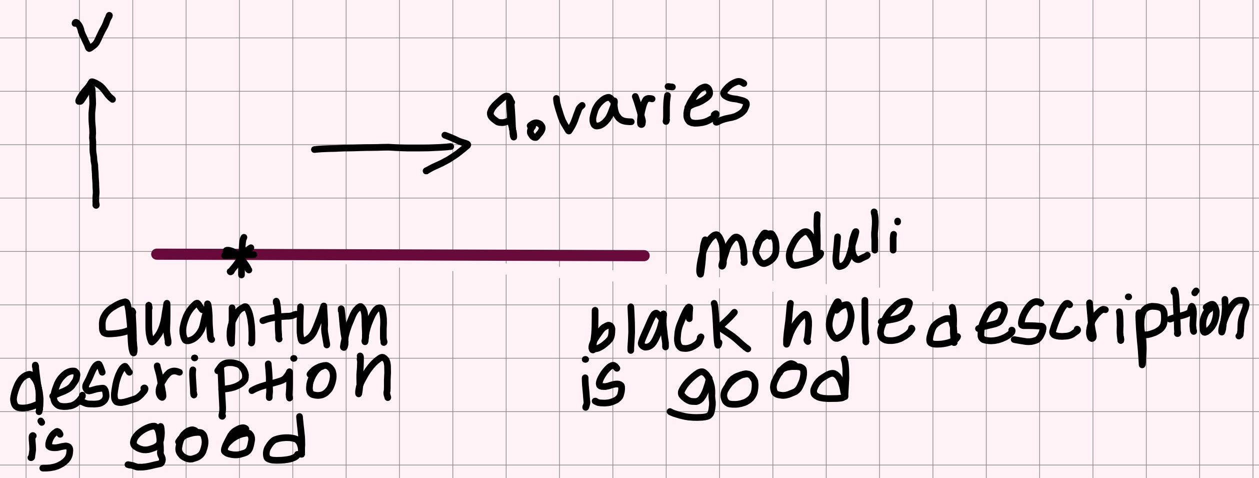
Nq_0 small $\rightarrow$ the system can be regarded as a weakly interacting system of N particles

each of charge q_B $\Rightarrow$

quantize & count number of states Ω

compare $S \leftrightarrow \ln \Omega$

moduli space



susy $\rightarrow$ Ω does not change along the moduli space

1) A generic string compactification with $N \geq 2$ susy has multiple vector fields
($U(1)$ gauge fields)

$\Rightarrow$ A state carries multiple charges

$$\begin{array}{cccc} Q_1, & Q_2, & \ldots, & Q_K \\ \| & \| & & \\ N_1 q_1^{(0)}, & N_2 q_2^{(0)}, & \ldots & N_K q_K^{(0)} \end{array}$$

2. (no. of bosonic states − no. of fermionic states)

compare $S = \frac{A}{4G} \longleftrightarrow \ln \Omega$

supersymmetry to the rescue!

susy $\Rightarrow$ black holes carry zero angular momentum

$\Rightarrow$ ALL microstates are bosonic when we have susy black holes

String theory in $D=10$ (IIA, IIB)

compactify 6 direction on product of 6 circles

$x^m \equiv x^m + 2\pi R_m$ for $m = 4,5,6,7,8,9$

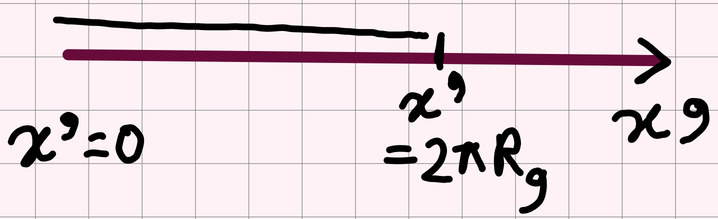
$x^\mu : \mu = 0,1,2,3$ non compact (3+1) dim space time coordinates

$x^0 \equiv t$

$$2\pi R_9 \equiv \int_0^{2\pi} \sqrt{g_{99}}\, dx^9 = 2\pi \sqrt{g_{99}}$$

This is the sense in which R_9 is a modulus.

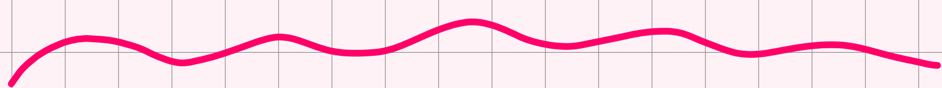
Take a fundamental string and wind it once around the 9th direction.



For R_9 large, the dynamics will be small oscillations in the $\perp$ direction

Degrees of freedom

$$X^i(x^9, t)$$



Action

$$\propto \int dt\, dx^9 \sum_{i=1}^{8} \left\{ (\partial_t X^i)^2 - (\partial_{x^9} X^i)^2 \right\}$$

$$\partial_{x^9} = \frac{\partial}{\partial x^9}$$

mode expansion

$$X^i(x^9, t) = \sum_{n \in \mathbb{Z}} X^i_n(t) e^{inx^9/R_9}$$

$$\text{Action} \propto \sum_{i=1}^{8} \sum_{n} \left(\partial_t X^i_{-n}\, \partial_t X^i_n - \frac{n^2}{R_9^2} X^i_{-n} X^i_n \right)$$

$$X^i_{-n} = (X^i_n)^*$$

compare with harmonic oscillator

$$\text{Action} \propto \frac{1}{2}(\dot{x}^2 - \omega^2 x^2) \qquad \text{h.o. } \omega = \frac{n}{R_9}$$

$$\text{States: } \prod_{i=1}^{8} \prod_{n} (a^\dagger_{i,n})^{r_{i,n}} |0\rangle \qquad [a_{i,n}, a^\dagger_{j,n'}] = \delta_{ij}\delta_{nn'}$$

$$\text{momentum} = \sum_i \sum_n \frac{n}{R_g} r_{i,n}$$

$$\text{Energy} = \text{const.} + \sum_i \sum_n \frac{|n|}{R_g} r_{i,n}$$

each $a^\dagger_{i,n}$ carries x^9 momentum $\frac{n}{R_g}$

g_{99} is a scalar

$g_{9\mu} \to$ vector

$\sum_{i=1}^{8} \sum_n \frac{n}{R_g} r_{i,n}$ is the charge of $g_{9\mu}$

extremal states

all +ven or all -ven

Suppose we are looking for States with Charge $\frac{N}{R_s}$ (N= integer)

$$N = \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} n\, r_{i,n}$$

$\Omega(N)$ = no of ways we can choose integer $r_{i,n} \geq 0$ such that

$$\sum_{i=1}^{8} \sum_{n=0}^{\infty} n r_{i,n} = N$$

$N = 1 \Rightarrow 8$

Partition function

$$f(q) = \sum_{N} \Omega(N) q^N$$

complex variable

$$f(q) = \sum_{N} q^N \prod_{i=1}^{8} \prod_{n=1}^{\infty} \prod_{r_{i,n}=0}^{\infty} \delta_{\sum_{i,n} n r_{i,n}, N}$$

$$\sum_{r_{1,1}=0}^{\infty} \sum_{r_{2,1}=0}^{\infty} \cdots \sum_{r_{8,1}=0}^{\infty} \sum_{r_{1,2}=0}^{\infty} \cdots \sum_{r_{8,2}=0}^{\infty} \cdots$$

$$= \prod_{i=1}^{8} \prod_{n=1}^{\infty} \sum_{r_{i,n}=0}^{\infty} q^{\sum_{i,n} n \, r_{i,n}}$$

$$= \prod_{i=1}^{8} \prod_{n=1}^{\infty} \sum_{r_{i,n}=0}^{\infty} q^{r_{i,n} \, n}$$

$$\longleftrightarrow \frac{1}{1-q^n}$$

$$f(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^8}$$

expand in power series in q.

Calculate $\Omega(N)$

Fit with $\Omega(N) = A\sqrt{N} + B\ln N + C + \dots$

calculate A, B, C for large N

reviews:

link 1

link 2

General case

∞ no. harmonic oscillators

i-th oscillator carries

charge $(n_1^{(i)} q_1^{(0)}, n_2^{(i)} q_2^{(0)}, \ldots n_K^{(i)} q_K^{(0)})$

in units of minimal charges.

$(N_1 q_1^{(0)}, N_2 q_2^{(0)}, \ldots N_K q_K^{(0)})$

minimal units of charge

Large integers

general state:

$\prod (a_i^\dagger)^{r_i} |0\rangle$, r_i integers $0 \le r_i < \infty$

charge:

$$\left(\sum_i r_i n_1^{(i)} q_1^{(0)}, \sum_i r_i n_2^{(i)} q_2^{(0)}, \ldots\right)$$

$$= (N_1 q_1^{(0)}, N_2 q_2^{(0)}, \ldots N_K q_K^{(0)})$$

$$N_1 = \sum_i r_i n_1^{(i)}, \quad N_2 = \sum_i r_i n_2^{(i)}, \dots$$

Q. how many ways can we choose r_i's such that

$$N_1 = \sum_i r_i n_1^{(i)}, \quad N_2 = \sum_i r_i n_2^{(i)}, \dots$$

$$\Omega(N_1, \dots N_k) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \delta_{\sum_i r_i n_1^{(i)}, N_1} \, \delta_{\sum_i r_i n_2^{(i)}, N_2} \cdots \delta_{\sum_i r_i n_k^{(i)}, N_k}$$

$$\left(\prod_{i=1}^{\infty} \sum_{r_i} \right)$$

partition function

$$f(q_1, \dots q_N) = \sum_{N_1, N_2 \dots N_k} \Omega(N_1, \dots N_k) q_1^{N_1} q_2^{N_2} \dots q_k^{N_k}$$

$$f(q_1, \dots q_k) = \sum_{N_1} \sum_{N_2} \sum_{N_k} \left(\prod_{i=1}^{\infty} \sum_{r_i} \right) \delta_{\sum r_i n_1^{(i)}, N_1}$$

$$\cdots \delta_{\sum_i r_i n_k^{(i)}, N_k} \, q_1^{N_1} \dots q_k^{N_k}$$

$$= \left(\prod_{i=1}^{\infty} \sum_{r_i}\right) q_1^{\sum_i r_i n_{(1)}^{i}} \cdots\cdots q_k^{\sum_i r_i n_i^{(k)}}$$

$$= \left(\prod_{i=1}^{\infty} \sum_{r_i}\right) \prod_{i=1}^{\infty} \left(q_1^{n_1(i)} q_2^{n_2(i)} \cdots q_k^{n_k(i)}\right)^{r_i}$$

$$= \prod_{i=1}^{\infty} \sum_{r_i=0}^{\infty} \left(q_1^{n_1(i)} \cdots q_k^{n_k(i)}\right)^{r_i}$$

$$= \prod_{i=1}^{\infty} \left(1 - q_1^{n_1(i)} \cdots\cdots q_k^{n_k(i)}\right)^{-1}$$

$$\boxed{\sum_{r=0}^{\infty} x^r = (1-x)^{-1}}$$

Fermionic oscillators : $b^+ b$

$\{b, b^+\} = 1, \quad \{b, b\} = 0$

$b^2 = 0 \quad (b^+)^2 = 0$

suppose the ith fermionic oscillator carries

$$(m_1^{(i)} q_1^{(0)}, m_2^{(i)} q_2^{(0)}, \ldots m_k^{(i)} q_k^{(0)})$$

state: $\prod_i (a_i^+)^{s_i} |0\rangle$ $\quad s_i = 0, 1,$ charge

$\sum_i S_i = \text{odd} \rightarrow$ fermionic states

$\sum_i S_i = \text{even} \rightarrow$ bosonic states

the calculation follows.

Fermionic contribution to the partition function

$$\prod_{i=1}^{\infty} \sum_{S_i=0}^{1} \left(q_1^{m_1^{(i)}} \cdots q_k^{m_k^{(i)}}\right)^{S_i}$$

$$= \prod_{i=1}^{\infty} \left(1 + q_1^{m_1^{(i)}} \cdots q_k^{m_k^{(i)}}\right)$$

For index:

$$\prod_{i=1}^{\infty} \left\{ \sum_{S_i=0}^{1} \left(q_1^{(m_1^{(i)})} \cdots q_k^{(m_k^{(i)})}\right)^{S_i} (-1)^{S_i} \right\}$$

$$= \prod_{i=1}^{\infty} \left(1 - q_1^{m_1^{(i)}} q_2^{m_2^{(i)}} \cdots q_k^{m_k^{(i)}}\right)$$

$$\boxed{(-1)^{\sum_i S_i} = \prod_i (-1)^{S_i}}$$

suppose we have calculated $f(q_1, \dots q_k)$

$$f(q_1, \dots q_k) = \sum_{N_1} \sum_{N_2} \dots \sum_{N_k} \Omega(N_1, \dots N_k) q_1^{N_1} \dots q_k^{N_k}$$

Q. Given f_1, how do we calculate Ω?

for Black hole system, we expect Ω to grow for large N.

define $\tau_1, \dots \tau_K$ such that

$$q_1 = e^{2\pi i \tau_1}, \quad q_2 = e^{2\pi i \tau_2}, \dots$$

$$\boxed{\begin{array}{l}\text{contribution from charge of the vaccum is } q_1^{c_1} q_2^{c_2} \dots q_k^{c_k} \\ \text{if } |0\rangle \text{ carries charge } (c_1, \dots c_k)\end{array}}$$

$$f(q_1, \dots q_k) = \sum_{N_1} \dots \sum_{N_K} \Omega(N_1, \dots N_2) \, e^{2\pi i \tau_1 N_1 + 2\pi i \tau_2 N_2 + 2\pi i \tau_K N_K}$$

$$\therefore \Omega(N_1, \dots N_K) = \int d\tau_1 \dots \int d\tau_K \, f(q_1, \dots q_K) \, e^{-2\pi i N_1 \tau_1} \dots e^{-2\pi i N_K \tau_K}$$

$$f(x) = \sum_N e^{2\pi i N x} g(N) \Rightarrow g(N) = \int_0^1 dx \, e^{-2\pi i N x} f(x)$$

$$\Omega(N_1, \dots N_K) = \int_{i\Lambda_1}^{i\Lambda_1+1} d\tau_1 \dots \int_{i\Lambda_K}^{i\Lambda_K+1} d\tau_K \, f(q_1, \dots q_K) \, e^{-2\pi i N_1 \tau_1} \dots e^{-2\pi i N_K \tau_K}$$

$f(q_1, \dots q_K)$ is well defined for $\mathrm{Im}\,\tau_i$ large for every i.

N_K = large +ve number.

$\tau_1 = i\Lambda_1 + x_1$ $\quad e^{2\pi i \tau_1 N_1} = e^{-2\pi \Lambda_1 N_1} e^{2\pi i N_1 x}$

$q_K \to$ arbitrary complex var

$$f(q_1, \dots q_N) = \sum_{N_1, \dots N_K} \Omega(N_1, \dots N_K) q_1^{N_1} \dots q_K^{N_K}$$

$$\Omega(N_1, \ldots N_K) = \int_{i\Lambda_1}^{i\Lambda_1+1} d\tau_1 \ldots \int_{i\Lambda_K}^{i\Lambda_K+1} d\tau_K$$

$$\underbrace{f(q_1, \ldots q_K)\, e^{-2\pi i \tau_1 N_1} \ldots e^{-2\pi i \tau_K N_K}}$$

evaluate this using saddle point methods

heterotic string theory in $D=10$ and compactify 6 directions on circles.

$$x^m \equiv x^m + 2\pi R_m \quad \text{for } m = 4, \ldots 9$$

x^μ : $\mu = 0, 1, 2, 3$: non-compact directions

In $D=10$, we have $G_{MN}, B_{MN}, \phi, \ldots$ massless fields

$G_{m\mu}, B_{m\mu}$ $\quad m = 4, \ldots 9$, $\quad \mu = 0, \ldots 3$ $\quad M, N = 0, \ldots 9$

consider states with following charges:

1) Q units of magnetic charges of $B_{4\mu}$

2) one unit of electric charge of $B_{5\mu}$

3) J units of magnetic charge of $B_{5\mu}$

4) n units of electric charge of $G_{5\mu}$

5) one unit of magnetic charge of $G_{4\mu}$

here Q, n, J are arbitrary integers

$$f(q_1,q_2,q_3)=\sum \Omega(n,Q,J)\,q_1^n q_2^Q q_3^J$$

Results:

$$f(q_1,q_2,q_3)=q_1^{-1}q_2^{-1}q_3^{-1}\prod_{\substack{k,l,j\in\mathbb{Z}\\ k,l\geq 0\\ \text{if } k=0,\ l=0,\\ \text{then } j<0}}\left(1-q_1^k q_2^l q_3^j\right)^{-c(4kl-j^2)}$$

c's are known coefficients defined via

ϑ = jacobi theta function

$$8\left[\frac{\vartheta_{00}(\tau,z)^2}{\vartheta_{00}(\tau,0)^2}+\frac{\vartheta_{01}(\tau,z)^2}{\vartheta_{01}(\tau,0)^2}+\frac{\vartheta_{10}(\tau,z)^2}{\vartheta_{10}(\tau,0)^2}\right.$$

$$=\sum_{n,j\in\mathbb{Z}} c(4n-j^2)\,e^{2\pi i n\tau+2\pi i j z}$$

$$\vartheta_{00}(\tau,z)=\sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau}\,e^{2\pi i n z}$$

$$\vartheta_{01}(\tau,t)=\vartheta_{00}\left(\tau,z+\tfrac{1}{2}\right)$$

$$\vartheta_{10}(\tau,z)=\vartheta_{00}\left(\tau,z+\tfrac{\tau}{2}\right)\times e^{\pi i\frac{\tau}{4}}e^{\pi i z}$$

→ for long n, Q, J } $\Omega(n, Q, J)$

$$= \pi \sqrt{4 Q_1 n - J^2}$$

+corrections

Agrees with $S_{B.H}$

Ω in heterotic string theory

Another example: Type IIA/IIB on T^6

Take a state carrying:

$$x^m = x^m + 2\pi R_m \quad \text{for } m = 4, \dots, 9$$

$$\mu = 0, \dots, 3 \quad \text{non-compact}$$

1) One unit of magnetic charge of $G_{5\mu}$

2) One unit of magnetic charge of $B_{5\mu}$

3) One unit of electric charge of $B_{4\mu}$

4) N unit of magnetic charge of $G_{4\mu}$

calculate $\Omega(\Sigma)$

Result:

$$\Omega(N) = -\hat{c}(4N)$$

$\hat{c}$ is defined via:

$$p^{-1}(1-p)^2 \prod_{n=1}^{\infty} \frac{(1-q^m p)^2 (1-q^m p^{-1})^2}{(1-q^m)^4}$$

$$= \sum_{\substack{k,l \\ k \geq 0}} \hat{c}(4k - l^2)\, q^k p^l$$

example:

Calculate $\hat{c}(4)$ $\quad \Omega(4) = -\hat{c}(4)$

Look for coefficients of $q^1 p^0 = q$

$$(p^{-1})(1-p)^2 \frac{(1-qp)^2(1-qp^{-1})^2}{(1-q)^4}$$

$$= p^{-1}(1-p)^2 \{1 - 2qp - 2qp^{-1} + 4q\}$$

$$= (p^{-1} - 2 + p)\{1 - 2qp - 2qp^{-1} + 4q\}$$

$$\simeq -2q - 8q - 2q$$

$$= -12q$$

for large N

$$\ln \Omega(N) = 2\pi\sqrt{N} - 2\ln N + \cdots$$

this result has to be Obtained by saddle point analysis of the fourier expansion.

exercise: find this formula by computing $\Omega(N)$ numerically up to $N=20$ and fitting

$$\Omega(N) = A\sqrt{N} + B\ln N + \cdots$$

In theories with $N \geq 2$ susy the lagrangian is compicated.

fields:

scalars $\{\phi_\alpha\}$ $\alpha = 1, \ldots, n_s$

vectors $\{A_\mu^{(i)}\}$ $i = 1, \ldots n_v$

$g_{\mu\nu}$

$$\mathcal{L}(\{\phi_\alpha\}, \{F_{\mu\nu}^{(i)}\}, g_{\mu\nu})$$

How do we find black hole solutions and their entropy?

Go back to RN solution:

metric = ds^2

$$= -\left(1-\frac{a}{R}\right)\left(1-\frac{b}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{a}{r}\right)\left(1-\frac{b}{r}\right)}$$

$$+ r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

extremal limit : $b \to a$

define $\lambda = \frac{a-b}{2}$, $\tau = \frac{\lambda t}{a^2}$

$$\rho = \frac{2r-a-b}{2\lambda}$$

rewrite the metric in ρ, τ coordinates with b replaced by $a+2\lambda$ at fixed ρ, τ, a

exercise:
show that in this limit

$$ds^2 = a^2\left(-(e^2-1)\,d\tau + \frac{de^2}{e^2-1}\right) + a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The two horizons are at $e = \pm 1$.

change coordinate : $\rho = \cosh\eta$

$$ds^2 = \underbrace{a^2\,(-\sinh^2\eta\, d\tau^2 + d\eta^2)}_{\text{AdS}_2 \text{ - two dimensional anti. de sitter space}} + \underbrace{a^2(d\theta^2 + \sin^2\theta\, d\phi^2)}_{\text{sphere}}$$

sphere:

$x^2+y^2+z^2=a^2$ embedded in euclidean 3d space.

claim:

$ds^2 = dx^2+dy^2 +dz^2$

$z = a\cos\theta$

$y = a\sin\theta \sin\phi$

$x = a\sin\theta \cos\phi$

AdS_2 metric is given by $x^2-y^2-z^2=-a^2$ in the 3-D spacetime.

change of variable with metric $ds^2 = dx^2 - dy^2 - dz^2$

$z = a\cosh\eta$, $y = a\sinh\eta \sinh\tau$, $x = a\sinh\eta \cosh\tau$

$$y^2 - x^2 = a^2 \sinh^2\eta\,(-1)$$
$$= -a^2 \sinh^2\eta$$

exercise: check that this gives AdS_2 metric.

result:

All spherically symmetric external blackhole have $AdS_2 \times S^2$ near horizon geometry.

2d- sphere has $SO(3)$ symmetry.

AdS_2 has $SO(1,2)$ symmetry.

suppose that we have a theory containing

i) scalars $\{\phi_\alpha\}$

ii) vectors $A_\mu^{(i)}$

iii) metric $g_{\mu\nu}$

$$\phi_\alpha(\theta, \phi, \rho, \tau)$$

on the extremal horizon: $\phi_\alpha = u_\alpha$

constant independent of ρ, τ, θ, ϕ.

$$ds^2 = \vartheta_1(-\sinh^2\eta\, d\tau^2 + d\eta^2) + \vartheta_2(d\theta^2 + \sin^2\theta\, d\phi).$$

$\vartheta_1, \vartheta_2 =$ constants

$$= \vartheta_1\left(\frac{de^2}{e^2-1} - (e^2-1)d\tau^2\right) + \vartheta_2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$F^{(i)}_{\mu\nu}$ has no mixed component.

$F_{\theta\phi}^{(i)} = p_i \sin\theta$ $\qquad p_i =$ constant

$F_{\eta\tau}^{(i)} = e_i \sinh\eta$ $\qquad e_i =$ constant

$\Downarrow$

$F_{e\tau}^{(i)} = e_i$

$F^{(i)}_{\theta\phi}\, d\theta \wedge d\phi$

$\propto \sin\theta\, d\theta\, d\phi$

Action $A = \int d^4x \sqrt{-\det g}\, \mathcal{L}$ $\rightarrow$ Lagrangian density $\rightarrow$ a scalar independent of θ, ϕ, ρ, τ

$= \int de\, d\tau\, d\theta\, d\phi\; \vartheta_1 \vartheta_2 \sin\theta\, \mathcal{L}$

$\mathcal{L}(\{u_\alpha\}, \{e_i\}, \{p_i\}, \vartheta_1, \vartheta_2)$

eq of motion: $\frac{\delta A}{\delta(\text{fields})} = 0$

$4\pi \int d\rho \, dr$

$$\underbrace{\vartheta_1 \vartheta_2 \mathcal{L}(\{u_\alpha\}, \{e_i\}, \{p_i\}, \vartheta_1, \vartheta_2)}_{f}$$

scalar field eqns:

$$\frac{\delta A}{\delta \phi_\alpha} = 0 \Rightarrow \frac{\partial f}{\partial u_\alpha} = 0$$

metric eq.

$$\frac{\partial f}{\partial \vartheta_1} = 0 \, , \quad \frac{\partial f}{\partial \vartheta_2} = 0$$

$$ds^2 = \vartheta_1 \left(\frac{de^2}{e^2 - 1} - (e^2 - 1) dr^2 \right) + \vartheta_2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

electromagnetic field equations:

$$D_\mu F^{(i)}_{\nu e} = 0$$

$$\frac{\delta A}{\delta A_\mu^{(i)}} = 0$$

$$\Rightarrow \frac{\partial}{\partial x^\nu}\left(\sqrt{-\det g}\,\frac{\delta \mathcal{L}}{\delta F_{\mu\nu}^{(i)}}\right) = 0 \Rightarrow \text{automatically satisfied}$$

due to the absence of invariant vector.

p_i = magnetic charge

q_i = electric charge $\quad q_i = \frac{\partial f}{\partial e_i}$

$f = v_1 v_2 \mathcal{L}$

result:

$$\text{entropy} = 8\pi^2\Big(\sum_i q_i e_i - f\Big)$$

e is an electric field.

near horizon geometry of extremal black holes

$$ds^2 = v_1\left(\frac{de^2}{e^2-1} - (e^2-1)\,d\tau^2\right) + v_2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$\phi_\alpha = u_\alpha, \quad F^{(i)}_{\theta\phi} = p_i \sin\theta, \quad F^{(i)}_{\rho\tau} = e^i$$

$$q_i = \frac{\partial}{\partial e_i}(v_1 v_2 \mathcal{L})$$

define a function

$$\mathcal{E}(\{u_\alpha\}, v_1, v_2, \{e_i\}, q_i, p_i) \equiv 8\pi^2\left(\sum_i q_i e_i - v_1 v_2 \mathcal{L}\right)$$

u_α, u_1, u_2, e_i are obtained from;

$$\frac{\partial \mathcal{E}}{\partial u_\alpha} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0$$

Entropy $= \mathcal{E}$ at the solution.

Test this in einstein-maxwell theory

$$\mathcal{L} = \left(\frac{1}{8\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

substitute the fields, set $p=0$ (for comparison with easier results)

$$R = 2\left(\theta_2^{-1} - \theta_1^{-1}\right), \quad \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} e^2 \theta_1^{-2}$$

$$\mathcal{E} = 8\pi^2 \left[qe - \frac{\theta_1 \theta_2}{16\pi G} \left(2\theta_2^{-1} - 2\theta_1^{-1} \right) - \frac{1}{2} e^2 \theta_1^{-1} \theta_2 \right]$$

$$\frac{\partial \mathcal{E}}{\partial \theta_1} = 0 \quad \Rightarrow -\frac{1}{8\pi G} + \frac{e^2}{2} \frac{\theta_2}{\theta_1^2} = 0$$

$$\frac{\partial \mathcal{E}}{\partial \theta_2} = 0 \quad \Rightarrow +\frac{1}{8\pi G} - \frac{e^2}{2} \theta_1^{-1} = 0$$

$$\frac{\partial \mathcal{E}}{\partial e} = 0 \Rightarrow q - e\theta_1^{-1}\theta_2 = 0$$

exercise:

$e = q, \quad \theta_1 = \theta_2 = 8\pi G q^2$

$S = \mathcal{E} = 4\pi^2 q^2$

check that these agree with earlier results.

Applying to heterotic on T^6 and type II on T^6.

relevant theory in both cases contain:

1) two scalars $a, \sigma \Rightarrow$ axion dilaton

2) scalars from G_{mn}, B_{mn}

$\Rightarrow$ can be written as 12×12 matrix $M(x)$ satisfying

$$M^T = M, \quad M L M^T = L, \quad L = \begin{pmatrix} 0 & I_{6\times 6} \\ I_{6\times 6} & 0 \end{pmatrix}$$

3) 12 vector fields $A_\mu^{(i)}$ $i = 1, \dots 12$

4) metric $G_{\mu\nu}$

$$\mathcal{L} = \frac{1}{2\pi\alpha'} \sigma \Big[R_G + \frac{1}{\sigma^2} \big(G^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma$$

$$- \frac{1}{2} G^{\mu\nu} \partial_\mu a \, \partial_\nu a \big) + \frac{1}{8} G^{\mu\nu} \mathrm{Tr} \left(\partial_\mu M L \partial_\nu M L \right)$$

$$- G^{\mu\nu} G^{\mu'\nu'} \sum_{i,j} (LML)_{ij} \, F^{(i)}_{\mu\mu'} F^{(j)}_{\nu\nu'}$$

$$- \frac{a}{\sigma} G^{\mu\mu'} G^{\nu\nu'} F_{\mu\mu'}^{(i)} L_{ij} \tilde{F}_{\nu\nu'}^{(j)} \Big]$$

near horizon fields

$\sigma = us$, $a = u_a$, $M_{ij} = \mathcal{U} M_{ij}$

$$ds^2 = \frac{\alpha'}{16} \Big[Q \left(\frac{de^2}{e^2 - 1} - (e^2 - 1) d\tau^2 \right)$$

$$+ Q_1 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \Big]$$

$$F^{(i)}_{\theta\phi} = \frac{\sqrt{\alpha'}}{4} p_i \sin\theta \,.\quad F^{(i)}_{p r} = \frac{\sqrt{\alpha'}}{4} e_i$$

$$\mathcal{E} = 8\pi^2 \left(\sum e_i q_i - \left(\frac{\alpha'}{16}\right)^2 \Theta_1 \Theta_2 \mathcal{L} \right)$$

$$= 8\pi^2 \Big[\sum e_i q_i - \frac{1}{8} \Theta_1 \Theta_2 u_5 \Big\{ -\frac{2}{\Theta_1} + \frac{2}{\Theta_2} + \frac{2}{\Theta_1^2} e_i (L U_m L)_{ij} e_j$$

we have to extremize

solve:

$$\frac{\partial \mathcal{E}}{\partial e_i} = 0, \quad \frac{\partial \mathcal{E}}{\partial \Theta_i} = 0 \quad i = 1, 2, \quad \frac{\partial \mathcal{E}}{\partial (U_m)_{ij}} = 0$$

$\mathcal{E}$ gives the entropy.

$M \underline{I} M, \quad M L M^T - L$

$U_m^T = U_m, \quad U_m L U_m^T = L$

the state for which the counting was done in heterotic theory:

$$q = \begin{pmatrix} 0_5 \\ -n \\ 0_5 \\ -1 \end{pmatrix}$$

$$p = \begin{pmatrix} 0_4 \\ 1 \\ 0 \\ 0_4 \\ Q-1 \\ -J \end{pmatrix}$$

result:

$$\pi \sqrt{4Qn - J^2}$$

for the type II black hole for which we gave the counting -

$$q = \frac{1}{8\pi}\begin{pmatrix} 0_5 \\ -n \\ 0_5 \\ -1 \end{pmatrix}, \quad \mathcal{E} \text{ at extremum} = \pi\sqrt{4N}$$

$$p = \begin{pmatrix} 0_4 \\ 1 \\ 0 \\ 0_5 \\ 1 \\ 0 \end{pmatrix}$$

In heterotic on T^6, we have corrects to $\mathcal{L}$.

$$\Delta\mathcal{L} = \chi(a,\sigma)\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

$$\boxed{\chi(a,\sigma) = -\frac{1}{64\pi^2}\left[12\ln\sigma + 24\ln\eta(a-i\sigma)\right]}$$

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right)$$

correction to the entropy

$$\Delta S = 64\pi^2 \chi\left(\frac{J}{2Q}, \frac{\sqrt{4Qn - J^2}}{2Q}\right)$$

same correction appears in the counting formula

$\rightarrow$ is also true in a wide class of theories.

some consequences

S (entropy) is the extremum of $\mathcal{E}(\{u_\alpha\}, \varphi_1, \varphi_2, \{e_i\}, \{q_i\}, \{p_i\})$ with respect to $u_\alpha, \varphi_1, \varphi_2, e_i$

$\Rightarrow$ function of $\{q_i\}, \{p_i\}$

Full black hole solution also depends on moduli (ϕ_α at ∞)

Independence of S of the moduli is known as attractor mechanism.

the black hole has some susy near horizon AdS_2 symmetry $SO(1,2)$

result:

the minimal group that accomodates both is $PSU(1,1|2)$

Bonic part is $SO(1,2) \times SU(2)$

rotation

solution is rotationally invariant

$\Rightarrow$ no angular momentum

$\Rightarrow$ only bosonic states

Quantum theory

(on the gravity side)

euclidean path integral: $\tau \to i\tau_E$

$e^{iA} \to e^{A}$ (weight in the path integral)

$\downarrow$

Action

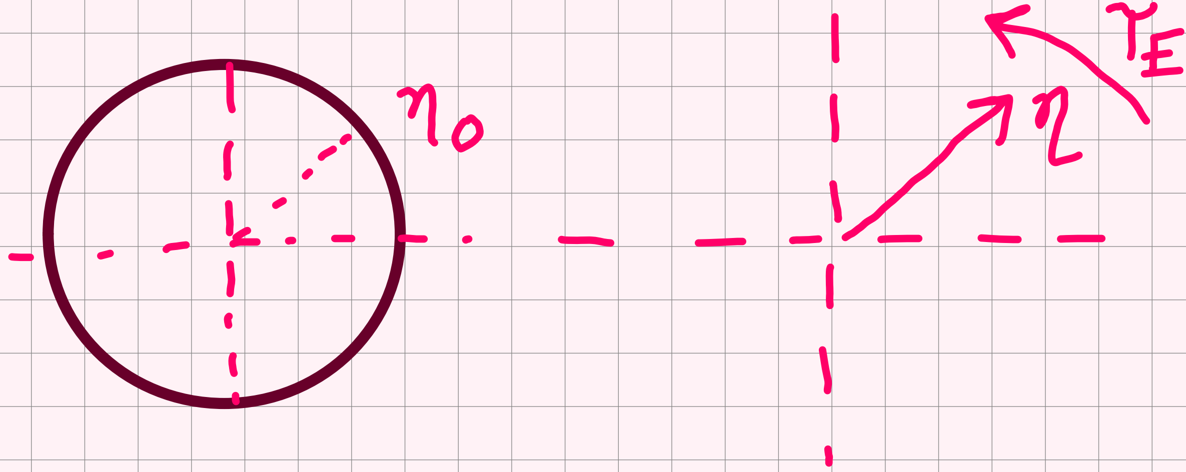
metric in polar coordinates

$dr^2 + r^2 d\theta^2$

eucidean solution

$$ds^2 = \Theta_1\left(\frac{de^2}{e^2-1} (e^2-1)d\tau_E^2\right) + \Theta_2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$= \underbrace{\Theta_1\left(d\eta^2 + \sinh^2\eta\, d\tau_E^2\right)}_{\text{smooth if } \tau_E \text{ has period } 2\pi} + \Theta_2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$



eucidean path integral

$$\int \mathcal{D}(\text{fields})\, e^{A} = f(\{q_i\}, \{p_i\})$$

Leading term:

$$e^{A}\Big|_{\text{classical soln}}$$
$$= e^{-8\pi^2\left(\sum_i q_i e_i \;-\; \theta_1\theta_2 \hbar\right)}$$

$\Downarrow$

classical result

the quantum fluctuations give term $\propto \ln(\text{Area})$

$\Rightarrow -2\ln N$ for IIA on T^6

0 for heterotic on T^6

agree with microscopic results

result:

so far the microscopic results & black hole results always agree.

notes by nazlee (or nafisa)